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REFINED THEORY OF COMPOSITE BEAMS: THE ROLE OF SHORT-WAVELENGTH EXTRAPOLATION

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Abstract—The present paper presents an asymptotically-correct beam theory with nonclassical sectional degrees of freedom. The basis for the theory is the variational–asymptotical method, a mathematical technique by which the three-dimensional analysis of composite beam deformation can be split into a linear, two-dimensional, cross-sectional analysis and a nonlinear, one-dimensional, beam analysis. The elastic constants used in the beam analysis are obtained from the cross-sectional analysis, which also yields approximate, closed-form expressions for three-dimensional distributions of displacement, strain, and stress. Such theories are known to be valid when a characteristic dimension of the cross section is small relative to the wavelength of the deformation. However, asymptotically-correct refined theories may differ according to how they are extrapolated into the short-wavelength regime. Thus, *there is no unique asymptotically-correct refined theory of higher order than classical (Euler–Bernoulli-like) theory*. Different short-wavelength extrapolations can be obtained by changing the meaning of the theory's one-dimensional variables. Numerical results for the stiffness constants of a refined beam theory and for deformations from the corresponding one-dimensional theory are presented. It is shown that a theory can be asymptotically correct and still have non-positive-definite strain energy density, which is completely inappropriate mathematically and physically. A refined beam theory, which appropriately possesses a positive-definite strain energy density and agrees quite well with experimental results, is constructed by using a certain short-wavelength extrapolation.

INTRODUCTION

The analysis of generally anisotropic beams has been the subject of many investigations, from both engineering and mechanics points of view, as reviewed by Hodges (1990a). Atılgan *et al.* (1991) made an attempt to develop a generally anisotropic beam theory based on the variational–asymptotical method of Berdichevsky (1979). The first (classical) approximation is developed without difficulty for composite beams having arbitrary layup, but the second approximation becomes intractable except in the isotropic case.

Because of the relatively large flexibility of composite beams in transverse shear one needs to incorporate shear deformation into the analysis. This was done for a linear theory by Giavotto *et al.* (1983) independent of asymptotical considerations; and thus, despite the generality of the analysis therein, the asymptotical accuracy of that theory is difficult to evaluate.

The approach of Hodges *et al.* (1992), intended to partially remedy this problem, was to incorporate transverse shear deformation directly into the first approximation. It was found that the strain energy could be minimized with respect to the transverse shear strain measures without loss of accuracy for long beams. For short beams (or for short wavelength deformation) there is still the need to develop a more refined theory. Unfortunately, when one attempts to extend the asymptotical validity of either Atılgan *et al.* (1991) or Hodges *et al.* (1992), one is faced with an interaction term in the strain energy very similar to that which is found in Atılgan and Hodges (1992) for laminated composite plates. Unless this troublesome term can be made to vanish rigorously, the resulting theory will not be asymptotically correct. A means for dealing with this problem has now been formulated [see Cesnik *et al.* (1993)] and it involves the solution of an eigenvalue problem over the cross-sectional plane of the beam. The eigenvectors associated with the zero eigenvalues

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correspond to the usual sectional degrees of freedom for classical beam theory—namely, three displacement components of the average reference line and one rotation about the average reference line. Additional eigenvectors will describe sectional deformation, such as that due to transverse shear, restrained torsional warping, etc. This approach guarantees that additional warping induced by these explicit degrees of freedom is of a higher order than the displacements associated with them.

In this paper we explore the anisotropic beam theory of Cesnik *et al.* (1993) by including a finite number of these additional degrees of freedom. In the contrast to the previous works [see for example Berdichevsky, 1979; Atılgan and Hodges, 1992] that use the asymptotical approach, we assumed that any new degrees of freedom have an order of magnitude that is of the order of the classical degrees of freedom. Indeed, the order of any degree of freedom is determined by the external loading and can be arbitrary.

There are two concepts to be considered when constructing a refined beam theory. To be of general usefulness, a theory must be asymptotically correct. An *asymptotically correct beam theory of a given order* is one for which the asymptotic expansion of the exact three-dimensional solution in terms of the identified small parameters coincides up to the given order with the final expansion obtained from the recovering relations [see Sutyrin and Hodges (1995)]. Kennard (1953) was probably the first to notice that there may be many differential equations which are asymptotically equivalent. The salient issue is that an asymptotically correct refined beam theory is not unique. In this connection an additional logically independent step should be made: short-wavelength extrapolation. The necessity of this step was recognized by Berdichevsky (1979) where the term "short-wavelength extrapolation" was introduced. This term comes from dynamics, but it is a convenient one to use in a context of static phenomena to refer to functions that rapidly change along the beam.

It is known that different short-wavelength extrapolations can be obtained by changing the meaning of the primary variables to be used in the theory, as in Berdichevsky (1973), Berdichevsky and Le (1982), and Atılgan and Hodges (1992). However, the question of how to generally implement short-wavelength extrapolation is still a subject for research. Our hypothesis is that it needs to be done by using changes of variable that are motivated by the necessity to match the dispersion curves associated with the one- and three-dimensional theories. Numerical experimentation is needed to confirm this (and is presently under investigation).

In what follows, we first provide a recapitulation of Cesnik *et al.* (1993), taking this opportunity to correct some typographical errors therein. In this context we also discuss the process of identifying the influence of the new degrees of freedom, the positive definiteness of the strain energy per unit length, and our proposed short-wavelength extrapolation procedure which involves changes of one-dimensional variables. We finally present numerical results for various types of beam cross sections and one-dimensional results.

ANALYTICAL PROCEDURE

The objective is to derive a strain energy functional of a beam in terms of one-dimensional quantities only. It can be done only if some small parameters are present. We suppose that the magnitude of strain is small compared to unity and that the cross section size is small relative to two other characteristic lengths: one over which the deformation varies and another which represents the magnitude of initial curvature and twist. The final result is specialized for prismatic composite beams.

Three-dimensional formulation

First, we derive a three-dimensional formulation, the solution of which shall be considered the exact solution of the beam problem.

Beam in the undeformed state. A typical point in the undeformed beam can be located by its arclength x along a reference line r and its cross-sectional local Cartesian coordinates $y \equiv \{y_2, y_3\}$ which vary in the prescribed domain S. The characteristic size of the domain

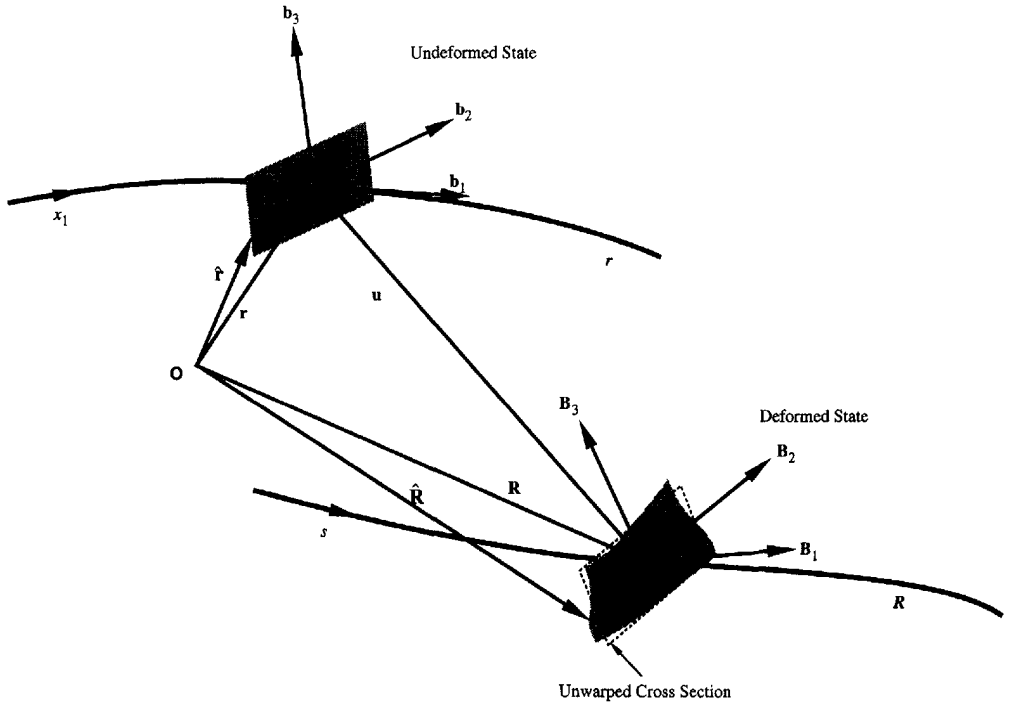

Fig. 1. Schematic of beam deformation (**u** is the displacement vector).

S is denoted by $h \equiv \sqrt{|S|}$, where $|S|$ is the area of S, and the dimensionless coordinates $\zeta \equiv \{\zeta_2 \equiv y_2/h, \zeta_3 \equiv y_3/h\}$ are introduced.

The spatial position vector $\hat{\mathbf{r}}$ to an arbitrary point in the undeformed beam can be written as

$$\hat{\mathbf{r}}(x, y_2, y_3) = \mathbf{r}(x) + y_\alpha \mathbf{b}_\alpha(x) \tag{1}$$

where $\mathbf{r}(x)$ is the spatial position vector of points on the undeformed reference line r, and the unit vectors $\mathbf{b}_\alpha(x)$ (Greek indices vary from 2 to 3, while Roman indices vary from 1 to 3; the repeated indices are summed over their range) are parallel to the planar Cartesian coordinate axes y_α in the plane of the reference cross section at a typical value of x. The three vectors $\mathbf{b}_n(x)$: $\{\mathbf{b}_1(x) \equiv \mathbf{b}(x), \mathbf{b}_2(x), \mathbf{b}_3(x)\}$ form an orthonormal triad with vector $\mathbf{b}$ being perpendicular to the reference cross-sectional plane and tangent to the reference line r.

The undeformed state also has several other important characteristics.

Let $\mathbf{k}$ [as in Danielson and Hodges (1987)] be the initial curvature vector. The expression for its components k_n is represented by

$$\mathbf{k} \cdot \mathbf{b}_n \equiv k_n = \tfrac{1}{2} e_{lmn} \mathbf{b}_l \cdot \mathbf{b}'_m. \tag{2}$$

Here and below, prime denotes differentiation with respect to x and e_{lmn} are Cartesian components of the permutation tensor.

The first component $k_1 \equiv k$ is the pretwist of the beam while k_α are components of the curvature of the reference line.

We will also need the contravariant base vectors ($\mathbf{g}^n$) defined by

$$\mathbf{g}^n(x, y) = \frac{1}{2\sqrt{g}} e_{lmn} \frac{\partial \hat{\mathbf{r}}}{\partial x_l} \times \frac{\partial \hat{\mathbf{r}}}{\partial x_m} \tag{3}$$

where the notation x_i, with $i = 1, 2, 3$, represents x, y_2, and y_3. The metric determinant g can be calculated as

$$\sqrt{g} = 1 - y_2 k_3 + y_3 k_2. \tag{4}$$

The reference line r is chosen in such a way that

$$\langle y_\alpha \rangle = \langle \zeta_\alpha \rangle = 0 \tag{5}$$

where the notation

$$\langle \bullet \rangle = \int_S \bullet \sqrt{g}\, \mathrm{d}y_2\, \mathrm{d}y_3 = h^2 \int \bullet \sqrt{g}\, \mathrm{d}\zeta_2\, \mathrm{d}\zeta_3 \tag{6}$$

is used throughout the paper.

Beam in the deformed state. Without any restrictions, the position vector $\hat{\mathbf{R}}$ which points to an arbitrary point in the deformed beam, can be represented by

$$\hat{\mathbf{R}}(x, y_2, y_3) = \mathbf{R}(x) + y_\alpha \mathbf{B}_\alpha(x) + v_n(\mathbf{x}, \zeta)\mathbf{B}_n(x) \tag{7}$$

where $\mathbf{R}(x)$ is the position vector of points on the deformed reference line R, and $\mathbf{B}_n(x)$: $\{\mathbf{B}_1(x) \equiv \mathbf{B}(x), \mathbf{B}_2(x), \mathbf{B}_3(x)\}$ is the reference orthonormal triad with $\mathbf{B}$ being tangent to the deformed reference line R. Elements $v_n(x, \zeta)$ are components of the general warping displacement of an arbitrary point in the deformed cross section, consisting of both in- and out-of-plane components, so that all possible deformations are considered.

The warping components $v_n(x, \zeta)$ cannot be defined as a function of ζ with an arbitrary choice of $\mathbf{B}_\alpha$, within a rotation around the vector $\mathbf{B}$, and with an arbitrary choice of $\mathbf{R}(x)$ unless they are subject to a suitable set of constraints, which may be chosen as

$$\begin{aligned} \langle v_n(x, \zeta) \rangle &= 0 \\ \langle \zeta_3 v_2(x, \zeta) - \zeta_2 v_3(x, \zeta) \rangle &= 0. \end{aligned} \tag{8}$$

Thus eqn (8) provides us with a convenient way of representing the arbitrary function $\hat{\mathbf{R}}(x, y_2, y_3)$. The orientation of the kinematical deformed beam triad $\mathbf{B}_n$ is now specified uniquely. It can be represented by an arbitrarily large rotation in terms of orientation angles, Rodrigues parameters, or any suitable angular displacement parameters. For additional discussion of this matter, see Hodges (1987).

Like the undeformed state, there is also the curvature vector $\mathbf{K}$ for the deformed state. The expression for its components K_n is represented by

$$\mathbf{K} \cdot \mathbf{B}_n \equiv K_n = \tfrac{1}{2} e_{lmn} \mathbf{B}_l \cdot \mathbf{B}'_m. \tag{9}$$

Strain field. The calculation of the strain field of the beam is based on the general formulation of Danielson and Hodges (1987). Under the condition of small local rotation, Jaumann strain components Γ^* (a 3×3 symmetric matrix) can be expressed by

$$\begin{aligned} \Gamma^* &= \tfrac{1}{2}(\chi + \chi^{\mathrm{T}}) - I \\ \chi_{mn} &= \mathbf{B}_m \cdot \frac{\partial \hat{\mathbf{R}}}{\partial x_k} \mathbf{g}^k \cdot \mathbf{b}_n \end{aligned} \tag{10}$$

where I is the 3×3 identity matrix.

Substituting eqn (7) in eqn (10), one can express the strain field as a 6×1 column matrix

$$\Gamma = \lfloor \Gamma^*_{11}\ 2\Gamma^*_{12}\ 2\Gamma^*_{13}\ \Gamma^*_{22}\ 2\Gamma^*_{23}\ \Gamma^*_{33} \rfloor^T$$

so that

$$\Gamma = \frac{1}{h}\Gamma_h v + \Gamma_\varepsilon \epsilon + \Gamma_R v + \Gamma_l v' \tag{11}$$

where matrices Γ_h (6×3), Γ_ε (6×4), Γ_R (6×3) and Γ_l (6×3) are

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial\zeta_2} & 0 & 0 \\ \frac{\partial}{\partial\zeta_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial\zeta_2} & 0 \\ 0 & \frac{\partial}{\partial\zeta_3} & \frac{\partial}{\partial\zeta_2} \\ 0 & 0 & \frac{\partial}{\partial\zeta_3} \end{bmatrix} \quad \Gamma_\varepsilon = \frac{1}{\sqrt{g}}\begin{bmatrix} 1 & 0 & \zeta_3 & -\zeta_2 \\ 0 & -\zeta_3 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_R = \frac{1}{\sqrt{g}}\begin{bmatrix} \tilde{k} + k_1 I\left(\zeta_3\frac{\partial}{\partial\zeta_2} - \zeta_2\frac{\partial}{\partial\zeta_3}\right) \\ 0 \end{bmatrix} \quad \Gamma_l = \frac{1}{\sqrt{g}}\begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{12}$$

Here $(\tilde{})_{nm} = -e_{nmk}(\ )_k$, and the column matrix ϵ represents the one-dimensional measures of deformation

$$\epsilon = \begin{Bmatrix} \gamma \\ h\kappa \end{Bmatrix} \tag{13}$$

where γ is the average cross-sectional extensional strain (the axial force strain measure) defined as

$$\gamma = \mathbf{R}' \cdot \mathbf{B} - 1 \tag{14}$$

and column matrix $\kappa = \lfloor \kappa_1 \kappa_2 \kappa_3 \rfloor^T$ contains the so-called moment strain measures

$$\kappa_n = K_n - k_n. \tag{15}$$

The small parameter ϵ can be now specified as

$$\epsilon = \max \|\epsilon\|. \tag{16}$$

A few nonlinear terms in the strain field, which couple v and ϵ, have been neglected in eqn (11) because a physically linear beam theory is to be developed. The form of the strain field is of great importance because it is now linear in ϵ, v and its derivatives. This is the only point where ε as a small parameter needs to be taken into account.

Strain energy of a beam. The strain energy density per unit length for a beam can be written as

$$U = \tfrac{1}{2}\langle \Gamma^{\mathrm{T}} D \Gamma \rangle \tag{17}$$

where D is the 6×6 symmetric material matrix in the $\mathbf{b}_n$ basis.

The three-dimensional Jaumann stress Z, which is conjugate to the Jaumann strain Γ is

$$Z = D\Gamma \tag{18}$$

Basic three-dimensional problem

The basic three-dimensional problem can be now represented as the following minimization problem

$$\int U\left[\epsilon(x), v(x,\zeta), \frac{\partial v(x,\zeta)}{\partial \zeta_\alpha}, v'(x,\zeta)\right] |S| \, \mathrm{d}x + \text{terms with external forces} \rightarrow \min \tag{19}$$

where the minimum should be found with respect to the arbitrary functions $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$, through which $\epsilon(x)$ is calculated, as well as with respect to the three functions $v_n(x,\zeta)$ which are subject to constraints defined in eqn (8).

Note that the arbitrary functions $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$ have only four degrees of freedom (three displacement and one in-plane rotation) due to the specific choice of the triad $\mathbf{B}_n$.

Dimensional reduction

Separation of the problem. Now the functional space of all the admitted functions $\hat{\mathbf{R}}(x,\zeta)$ is separated into subproblems with a choice of the x-dependent functions $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$. In each subproblem, the functions $v_n(x,\zeta)$ are arbitrary under the constraints from eqn (8).

We can solve this problem in two steps. First we find functions $v_n(x,\zeta)$ for any prescribed choice of functions $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$. As a result, we will have functions $v_n(x,\zeta)$ as a functional of $\mathbf{R}(x)$, $\mathbf{B}_n(x)$ and ζ, and the functional in eqn (19) will become dependent only on $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$. That functional will give us a one-dimensional beam problem. The second step would be to solve this problem.

Since the energy density U depends not only on the functions $v_n(x,\zeta)$ but also on their derivatives with respect to x, it is clear that the result of the first step will be too complicated. Indeed it will contain a non-local dependence on $\mathbf{R}(x)$ and $\mathbf{B}_n(x)$ in the general case and cannot be obtained in an appropriate form unless we take advantage of some small parameters.

Small parameters. There are four characteristic parameters in the considered theory, two of which, h and ε, have already been introduced. Two others are the characteristic length l, over which the deformation state varies in the longitudinal direction, and the characteristic length of the initial curvature and twist $R = 1/\max \|\mathbf{k}\|$.

Let us consider the situation in which the parameters h, l, R and ε are present. It is clear that the first term of eqn (11) has order $\|v\|/h$, the second has order ε, the third has order $\|v\|/R$ and the last has order $\|v\|/l$. The fourth term has order h/l times that of the first. We should neglect this as a higher order term in the first approximation if we intend to expand the solution with respect to the small parameter h/l. This important circumstance will allow us to avoid the presence of derivatives of the unknown functions $v_n(x,\zeta)$ with respect to x for any approximation, and then to solve it in an appropriate form. The third term has order h/R times that of the first. We should also neglect this as a higher order term in the first approximation. Note that the parameter h/R is also present inside some matrices Γ defined in eqn (12). This may be disregarded, as we do not need an additional expansion of h/R. The parameter ε does not need to be considered small any more, since our main problem has become linear with respect to the unknown functions $v_n(x,\zeta)$ and

the one-dimensional strain measure ε. As a small parameter, ε has already been taken into account (see subsection *Strain field* above).

We will expand the warping $v_n(x, \zeta)$ as a series with respect to the small parameters h/l and h/R. Since both of them have the same numerator, expansion in h/l and h/R is the same as the expansion in h only. We can therefore consider h to be the only small parameter in spite of its dimension.

Discretization

The problem may be solved numerically by discretizing it with respect to the cross-sectional coordinates ζ_α. Considering the finite element discretization, the unknown functions $v_n(x, \zeta)$ can be represented as the product of a shape functions matrix $S(\zeta)$ and a column matrix of nodal values of $v(x, \zeta)$, denoted as V

$$v(x, \zeta) = S(\zeta)V(x). \tag{20}$$

Substituting the above discretized unknown function into eqn (17) and also taking into account eqn (11), one obtains

$$2U = \left(\frac{1}{h}\right)^2 V^{\mathrm{T}}EV + \left(\frac{1}{h}\right)2V^{\mathrm{T}}(D_{h\varepsilon}\epsilon + D_{hR}V + D_{hl}V') + (1)(\epsilon^{\mathrm{T}}D_{\varepsilon\varepsilon}\epsilon + V^{\mathrm{T}}D_{RR}V + V'^{\mathrm{T}}D_{ll}V' + 2V^{\mathrm{T}}D_{R\varepsilon}\epsilon + 2V'^{\mathrm{T}}D_{l\varepsilon}\epsilon + 2V^{\mathrm{T}}D_{Rl}V') \tag{21}$$

in which the following definitions were introduced

$$\begin{aligned} E &\triangleq \langle [\Gamma_h S]^{\mathrm{T}} D [\Gamma_h S] \rangle & D_{\varepsilon\varepsilon} &\triangleq \langle [\Gamma_\varepsilon]^{\mathrm{T}} D [\Gamma_\varepsilon] \rangle \\ D_{h\varepsilon} &\triangleq \langle [\Gamma_h S]^{\mathrm{T}} D [\Gamma_\varepsilon] \rangle & D_{hR} &\triangleq \langle [\Gamma_h S]^{\mathrm{T}} D [\Gamma_R S] \rangle \\ D_{hl} &\triangleq \langle [\Gamma_h S]^{\mathrm{T}} D [\Gamma_l S] \rangle & D_{R\varepsilon} &\triangleq \langle [\Gamma_R S]^{\mathrm{T}} D [\Gamma_\varepsilon] \rangle \\ D_{l\varepsilon} &\triangleq \langle [\Gamma_l S]^{\mathrm{T}} D [\Gamma_\varepsilon] \rangle & D_{RR} &\triangleq \langle [\Gamma_R S]^{\mathrm{T}} D [\Gamma_R S] \rangle \\ D_{Rl} &\triangleq \langle [\Gamma_R S]^{\mathrm{T}} D [\Gamma_l S] \rangle & D_{ll} &\triangleq \langle [\Gamma_l S]^{\mathrm{T}} D [\Gamma_l S] \rangle. \end{aligned} \tag{22}$$

Classical considerations

According to the variational-asymptotical procedure, in order to get the next approximation, one should retain only the leading terms in the energy expression. These "leading" terms are with respect to the small parameter that contains the unknown functions and the leading intersection terms between the unknown function and the rest of the functional (for more details see Berdichevsky, 1979).

We are then left with the following expression

$$2U = \left(\frac{1}{h}\right)^2 V^{\mathrm{T}}EV + \left(\frac{1}{h}\right)2V^{\mathrm{T}}D_{h\varepsilon}\epsilon. \tag{23}$$

This functional must be minimized with respect to variable V under constraints

$$V^{\mathrm{T}}H\Psi_{cl} = 0 \tag{24}$$

where

$$H \triangleq \langle S^{\mathrm{T}}S \rangle \tag{25}$$

and Ψ_{cl} is a matrix with four columns, each corresponding to one of the constraints defined

in Cesnik *et al.* (1993). The set of columns Ψ_{cl} is determined by the kernel (null-space) of the matrix E. This implies

$$E\Psi_{cl} = 0. \tag{26}$$

Let us suppose that the set of columns Ψ_{cl} is normalized in such a way that

$$\Psi_{cl}^{\mathrm{T}} H \Psi_{cl} = I. \tag{27}$$

The Euler equation for the minimization problem defined by eqns (23) and (24) is given by

$$\left(\frac{1}{h}\right) EV + D_{h\varepsilon}\epsilon = H\Psi_{cl}\mu \tag{28}$$

where μ is the column matrix of Lagrange multipliers associated with eqn (24). By pre-multiplying eqn (28) by Ψ_{cl}^{T}, one can prove that

$$\mu = \Psi_{cl}^{\mathrm{T}} D_{h\varepsilon}\epsilon. \tag{29}$$

Subsequently, eqn (28) can be rewritten as

$$\left(\frac{1}{h}\right) EV = -(I - H\Psi_{cl}\Psi_{cl}^{\mathrm{T}}) D_{h\varepsilon}\epsilon. \tag{30}$$

The matrix E possesses a zero eigenvalue of multiplicity four, and thus its inverse does not exist. However, let us introduce the matrix E_{cl}^{+} with the following properties

$$\begin{aligned} EE_{cl}^{+} &= I - H\Psi_{cl}\Psi_{cl}^{\mathrm{T}} \\ E_{cl}^{+}E &= I - \Psi_{cl}\Psi_{cl}^{\mathrm{T}} H \\ E_{cl}^{+}EE_{cl}^{+} &= E_{cl}^{+} \end{aligned} \tag{31}$$

which is detailed in the Appendix. The solution of eqn (30) is then represented by

$$V = -hE_{cl}^{+} D_{h\varepsilon}\epsilon. \tag{32}$$

Substituting the above solution into the discretized strain energy density [eqn (21)] and keeping only terms with the lowest order, which are equal to $h^0 \equiv 1$, one obtains

$$2U = \epsilon^{\mathrm{T}} A_{cl}^{0} \epsilon \tag{33}$$

with

$$A_{cl}^{0} \triangleq D_{\varepsilon\varepsilon} - [D_{h\varepsilon}]^{\mathrm{T}} E_{cl}^{+} [D_{h\varepsilon}] \tag{34}$$

which is the classical result for the beam energy. Note that the third property from eqn (31) is taken into account here. It can be shown that the matrix A_{cl}^{0} is positive definite.

New degrees of freedom

In order to make our beam functional more flexible with respect to the variable x, let us introduce new unknown beam functions such that the new expression for V defined in eqn (21) is

$$V(x) = \Psi_q q(x) + W(x) \tag{35}$$

where q is a column matrix of several new unknown functions, and Ψ_q is a matrix, each column of which represents a ζ-mode shape of one of the new unknown functions $q(x)$. We denote these as the "new degrees of freedom". The new warping to be determined is W.

We assume that the matrix of the cross-sectional mode shapes, Ψ_q, is normalized in such a way that

$$\Psi_q^{\mathrm{T}} H \Psi_q = I. \tag{36}$$

The following constraint for W will make the separation in eqn (35) unique

$$W^{\mathrm{T}} H \Psi_q = 0. \tag{37}$$

The order of functions $q(x)$ with respect to h may be arbitrary and will be chosen here to be $h^0 \equiv 1$.

Let us take Ψ_q as being the eigenvectors of the matrix E which correspond to the lowest eigenvalues. Such a matrix Ψ_q will satisfy the following equation

$$E\Psi_q = H\Psi_q \Lambda_q \tag{38}$$

where Λ_q is a diagonal matrix of eigenvalues

$$\Lambda_q = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N_q} \end{bmatrix}. \tag{39}$$

The classical constraints [see Cesnik (1994)], which still might be satisfied by W, can be combined with constraints from eqn (37) which result in

$$W^{\mathrm{T}} H \Psi_u = 0 \tag{40}$$

where $\Psi_u = [\Psi_{cl} \Psi_q]$

Analogously, eqn (38) can be rewritten as

$$E\Psi_u = H\Psi_u \Lambda_u \tag{41}$$

where the matrix Λ_u is

$$\Lambda_u = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda_{N_q} \end{bmatrix}. \tag{42}$$

Energy calculation

Let us assume that we have the correct expansion of V through order h^2

$$V = V_0 + hV_1 + h^2 V_2 \tag{43}$$

where V_0 denotes the first term of eqn (35)

$$V_0 = \Psi_q q. \tag{44}$$

The column matrix V_0 satisfies the equation

$$EV_0 = H\Psi_q \Lambda_q q \tag{45}$$

and V_1 and V_2 must satisfy the eqn (40).

If we have an asymptotically correct expansion in eqn (43), we can calculate the asymptotically correct energy of order $h^0 \equiv 1$

$$\begin{aligned} 2U = &\left(\frac{1}{h}\right)^2 V_0^{\mathrm{T}} E V_0 + \left(\frac{1}{h}\right) 2V_0^{\mathrm{T}} (\underline{EV_1} + D_{h\varepsilon}\epsilon + D_{hR}V_0 + D_{hl}V_{0,x}) \\ &+ (1)[V_1^{\mathrm{T}} E V_1 + 2\underline{V_2^{\mathrm{T}} E V_0} \\ &+ 2V_1^{\mathrm{T}}(D_{h\varepsilon}\epsilon + D_{hR}V_0 + D_{hl}V_{0,x}) + 2V_{1,x}^{\mathrm{T}} D_{hl}^{\mathrm{T}} V_0 \\ &+ \epsilon^{\mathrm{T}} D_{\varepsilon\varepsilon}\epsilon + V_0^{\mathrm{T}} D_{RR} V_0 + V_{0,x}^{\mathrm{T}} D_{ll} V_{0,x} \\ &+ 2V_0^{\mathrm{T}} D_{R\varepsilon}\epsilon + 2\epsilon D_{\varepsilon l} V_{0,x}^{\mathrm{T}} + 2V_0^{\mathrm{T}} D_{Rl} V_{0,x}]. \end{aligned} \tag{46}$$

The underlined terms are equal to zero because of eqn (45) for V_0 and eqn (40) for V_1 and V_2. This means that we do not need to know the second approximation for V in order to calculate the energy of the order h^0. We shall minimize the functional

$$2U_0 = V_1^{\mathrm{T}} E V_1 + 2V_1^{\mathrm{T}}[D_{h\varepsilon}\epsilon + (D_{hR} + D_{hR}^{\mathrm{T}})V_0 + (D_{hl} - D_{hl}^{\mathrm{T}})V_{0,x}] \tag{47}$$

in order to find V_1. The term D_{hl}^{T} comes from the third line of eqn (46) after integration by parts with respect to x.

The Euler equation from eqn (47) is

$$EV_1 + D_{h\varepsilon}\epsilon + (D_{hR} + D_{hR}^{\mathrm{T}})V_0 + (D_{hl} - D_{hl}^{\mathrm{T}})V_{0,x} = H\Psi_u \mu_u \tag{48}$$

where μ_u is the Lagrange multiplier used to enforce eqn (40).

Applying a procedure similar to the one used for the classical case, we can calculate the Lagrange multiplier μ_u as

$$\mu_u = \Psi_u^{\mathrm{T}}[D_{h\varepsilon}\epsilon + (D_{hR} + D_{hR}^{\mathrm{T}})V_0 + (D_{hl} - D_{hl}^{\mathrm{T}})V_{0,x}]. \tag{49}$$

Using the above result in eqn (48), we finally get

$$EV_1 = -(I - H\Psi_u\Psi_u^{\mathrm{T}})[D_{h\varepsilon}\epsilon + (D_{hR} + D_{hR}^{\mathrm{T}})V_0 + (D_{hl} - D_{hl}^{\mathrm{T}})V_{0,x}]. \tag{50}$$

The solution for V_1 can be represented by

$$V_1 = -E_u^+[D_{h\varepsilon}\epsilon + (D_{hR} + D_{hR}^{\mathrm{T}})V_0 + (D_{hl} - D_{hl}^{\mathrm{T}})V_{0,x}] \tag{51}$$

where the matrix E_u^+ has the following properties

$$\begin{aligned} EE_u^+ &= I - H\Psi_u\Psi_u^{\mathrm{T}} \\ E_u^+ E &= I - \Psi_u\Psi_u^{\mathrm{T}} H \\ E_u^+ E E_u^+ &= E_u^+ \end{aligned} \tag{52}$$

(see the Appendix for an overview on how to calculate the matrix E_u^+).

Substituting the above expression for V_1 into the strain energy, we get

$$2U = \left(\frac{1}{h}\right)^2 V_0^{\mathrm{T}} E V_0 + \left(\frac{1}{h}\right) 2V_0^{\mathrm{T}}[D_{h\varepsilon}\epsilon + D_{hR}V_0 + D_{hl}V_{0,x}]$$
$$+ (1)(\epsilon^{\mathrm{T}} A_{\varepsilon\varepsilon}\epsilon + V_0^{\mathrm{T}} P_{RR} V_0 + V_{0,x}^{\mathrm{T}} P_{ll} V_{0,x} + 2V_0^{\mathrm{T}} P_{R\varepsilon}\epsilon + 2\epsilon^{\mathrm{T}} P_{\varepsilon l} V_{0,x} + 2V_0^{\mathrm{T}} P_{Rl} V_{0,x}) \quad (53)$$

where

$$\begin{aligned}
A_{\varepsilon\varepsilon} &\triangleq D_{\varepsilon\varepsilon} - [D_{h\varepsilon}]^{\mathrm{T}} E_u^{+} [D_{h\varepsilon}] \\
P_{RR} &\triangleq D_{RR} - [D_{hR} + D_{hR}^{\mathrm{T}}]^{\mathrm{T}} E_u^{+} [D_{hR} + D_{hR}^{\mathrm{T}}] \\
P_{ll} &\triangleq D_{ll} - [D_{hl} - D_{hl}^{\mathrm{T}}]^{\mathrm{T}} E_u^{+} [D_{hl} - D_{hl}^{\mathrm{T}}] \\
P_{R\varepsilon} &\triangleq D_{R\varepsilon} - [D_{hR} + D_{hR}^{\mathrm{T}}]^{\mathrm{T}} E_u^{+} [D_{h\varepsilon}] \\
P_{\varepsilon l} &\triangleq D_{\varepsilon l} - [D_{h\varepsilon}]^{\mathrm{T}} E_u^{+} [D_{hl} - D_{hl}^{\mathrm{T}}] \\
P_{Rl} &\triangleq D_{Rl} - [D_{hR} + D_{hR}^{\mathrm{T}}]^{\mathrm{T}} E_u^{+} [D_{hl} - D_{hl}^{\mathrm{T}}].
\end{aligned} \quad (54)$$

Finally, after using eqn (44), the beam energy density can be written as

$$2U = \begin{Bmatrix} \epsilon \\ \frac{1}{h} q \\ q, x \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{\varepsilon\varepsilon} & A_{\varepsilon q} & A_{\varepsilon l} \\ A_{q\varepsilon} & A_{qq} & A_{ql} \\ A_{l\varepsilon} & A_{lq} & A_{ll} \end{bmatrix} \begin{Bmatrix} \epsilon \\ \frac{1}{h} q \\ q, x \end{Bmatrix} \quad (55)$$

where

$$\begin{aligned}
A_{qq} &\triangleq \Psi_q^{\mathrm{T}}[E + h(D_{hR} + D_{hR}^{\mathrm{T}}) + h^2 P_{RR}]\Psi_q \\
&= \Lambda_q + \Psi_q^{\mathrm{T}}[h(D_{hR} + D_{hR}^{\mathrm{T}}) + h^2 P_{RR}]\Psi_q \\
A_{ql} &\triangleq \Psi_q^{\mathrm{T}}(D_{hl} + hP_{Rl})\Psi_q = A_{lq}^{\mathrm{T}} \\
A_{q\varepsilon} &\triangleq \Psi_q^{\mathrm{T}} D_{h\varepsilon} + h\Psi_q^{\mathrm{T}} P_{R\varepsilon} = A_{\varepsilon q}^{\mathrm{T}} \\
A_{ll} &\triangleq \Psi_q^{\mathrm{T}} P_{ll} \Psi_q \\
A_{\varepsilon l} &\triangleq P_{\varepsilon l}\Psi_q = A_{l\varepsilon}^{\mathrm{T}}.
\end{aligned} \quad (56)$$

The classical beam energy can be obtained from eqn (55) by minimizing it with respect to the variables q with the derivatives $q_{,x}$ being set equal to zero.

Finally, eqn (55) can be specialized for prismatic beams by removing all h/R effects from eqn (56). The prismatic stiffness submatrices associated with eqn (55) are

$$\begin{aligned}
A_{\varepsilon\varepsilon} &\triangleq D_{\varepsilon\varepsilon} - [D_{h\varepsilon}]^{\mathrm{T}} E_u^{+} [D_{h\varepsilon}] \\
A_{qq} &\triangleq \Psi_q^{\mathrm{T}} E \Psi_q = \Lambda_q \\
A_{ql} &\triangleq \Psi_q^{\mathrm{T}} D_{hl} \Psi_q = A_{lq}^{\mathrm{T}} \\
A_{q\varepsilon} &\triangleq \Psi_q^{\mathrm{T}} D_{h\varepsilon} = A_{\varepsilon q}^{\mathrm{T}} \\
A_{ll} &\triangleq \Psi_q^{\mathrm{T}} P_{ll} \Psi_q \\
A_{\varepsilon l} &\triangleq P_{\varepsilon l}\Psi_q = A_{l\varepsilon}^{\mathrm{T}}.
\end{aligned} \quad (57)$$

Influence of the q*s*

Due to the definition of the new degrees of freedom, all possible contributions of the variables q to the classical one-dimensional strain measures come from the interaction term $A_{q\varepsilon}$. The magnitude of the new degrees of freedom can be estimated via the classical strain measures. This can be done by setting $q_{,x}$ equal to zero and minimizing the remaining part of the strain energy with respect to q. The results are given by a relation of the form

$$q = \mathscr{I}_{q\varepsilon}\epsilon \tag{58}$$

where $\mathscr{I}_{q\varepsilon}$ is the "influence coefficient matrix" and is defined as

$$\mathscr{I}_{q\varepsilon} = A_{qq}^{-1}A_{q\varepsilon}. \tag{59}$$

One can judge the importance of a new degree of freedom by considering its corresponding influence coefficient. The larger the influence coefficient, the more important the corresponding new degree of freedom is.

The positive definiteness issue

It is important to notice that, in contrast to classical theory, no proof of positive definiteness of energy can be given for the stiffness matrix with the new degrees of freedom [eqn (55)]. To understand this, one can transform eqn (55) into the form

$$\begin{aligned} 2U = \frac{1}{h^2}&[q+h(A_{qq}^{-1}A_{q\varepsilon})\epsilon + hA_{qq}^{-1}A_{ql}q_{,x}]^{\mathrm{T}}A_{qq}[q+h(A_{qq}^{-1}A_{q\varepsilon})\epsilon + hA_{qq}^{-1}A_{ql}q_{,x}] \\ &+[\epsilon + A_{cl}^{-1}(A_{\varepsilon l}-A_{\varepsilon q}A_{qq}^{-1}A_{ql})q_{,x}]^{\mathrm{T}}A_{cl}[\epsilon + A_{cl}^{-1}(A_{\varepsilon l}-A_{\varepsilon q}A_{qq}^{-1}A_{ql})q_{,x}] \\ &+q_{,x}^{\mathrm{T}}[A_{ll}-A_{ql}^{\mathrm{T}}A_{qq}^{-1}A_{ql}+-(A_{\varepsilon l}-A_{\varepsilon q}A_{qq}^{-1}A_{ql})^{\mathrm{T}}A_{cl}^{-1}(A_{\varepsilon l}-A_{\varepsilon q}A_{qq}^{-1}A_{ql})]q_{,x} \end{aligned} \tag{60}$$

where A_{cl} is given by

$$A_{cl} = A_{\varepsilon\varepsilon}-A_{\varepsilon q}^{\mathrm{T}}A_{qq}^{-1}A_{\varepsilon q}. \tag{61}$$

This expression coincides with eqn (34) for a prismatic beam, and it is positive definite. If h/R is large enough the A_{cl} can be non-positive definite. This actually determines another upper bound of the applicable h/R. In other words A_{cl} is positive definite for initially twisted and curved beams as long as h/R is not too large.

The matrix A_{qq} is always positive definite by definition. Therefore, the last summand in eqn (60), which contains only $q_{,x}$ could be indefinite. And this indeed happens for some choices of the new degrees of freedom.

Also, the influence coefficients are present in eqn (60). It can be seen that the larger the influence coefficient is, the more negative indefinite the summand. This has been seen in our numerical tests, where it often happens that taking into account the most influential new degrees of freedom leads to a non-positive-definite energy density.

The appearance of this problem is natural. It is similar to losing positive definiteness in the expansion of the expression, $(1+\varepsilon)^2$, up to order 1 with respect to small ε. The classical theory, by which we denote the theory obtained from calculation of the leading terms in the asymptotic sense, is unique and does not have such a problem. Short-wavelength extrapolation is one possible way to correct this problem for refined theories.

Short-wavelength extrapolation

As pointed out above, the question of how to implement short-wavelength extrapolation is still open to investigation. Different short-wavelength extrapolations can be obtained by changing the meaning of the primary variables (one-dimensional in the case of beam theory). This was implemented in several papers, such as Berdichevsky (1973), Berdichevsky and Le (1982), and Atılgan and Hodges (1992), for example, for different

refined plate and beam, static and dynamic theories. Our hypothesis is that it needs to be done by using changes of variable to make one-dimensional and three-dimensional dispersion curves agree as closely as possible. This needs to be confirmed by numerical experimentation.

All possible short-wavelength extrapolations for a given choice of new degrees of freedom can be obtained by the following changes of variable

$$\begin{aligned} \epsilon &\to \epsilon + h\alpha q_{,x} \\ q &\to q + h\beta\epsilon \end{aligned} \tag{62}$$

where α and β are arbitrary matrices of appropriate dimensions. Substituting eqn (62) into eqn (55) and dropping out high-order terms, one obtains a transformation of the stiffness coefficients of the form

$$\begin{aligned} &A_{\varepsilon\varepsilon} \to A_{\varepsilon\varepsilon} + 2A_{\varepsilon q}\beta + \beta^{\mathrm{T}} A_{qq}\beta \quad && A_{\varepsilon q} \to A_{\varepsilon q} + \beta^{\mathrm{T}} A_{qq} \\ &A_{\varepsilon l} \to A_{\varepsilon l} + 2\beta^{\mathrm{T}} A_{ql} && A_{qq} \to A_{qq} \\ &A_{ql} \to A_{ql} && A_{ll} \to A_{ll} - 2\alpha^{\mathrm{T}} A_{\varepsilon q}. \end{aligned} \tag{63}$$

In order to ascertain if the energy density is positive definite or not, one can write the new expression for energy density in the form

$$\begin{aligned} 2U = \frac{1}{h^2}&[q + h(A_{qq}^{-1}A_{q\varepsilon} + \beta)\epsilon + hA_{qq}^{-1}A_{ql}q_{,x}]^{\mathrm{T}} A_{qq} \\ &\times [q + h(A_{qq}^{-1}A_{q\varepsilon} + \beta)\epsilon + hA_{qq}^{-1}A_{ql}q_{,x}] \\ &+ [\epsilon + A_{cl}^{-1}(A_{\varepsilon l} - A_{\varepsilon q}A_{qq}^{-1}A_{ql} + \beta^{\mathrm{T}}A_{ql})q_{,x}]^{\mathrm{T}} A_{cl} \\ &\times [\epsilon + A_{cl}^{-1}(A_{\varepsilon l} - A_{\varepsilon q}A_{qq}^{-1}A_{ql} + \beta^{\mathrm{T}}A_{ql})q_{,x}] \\ &+ q_{,x}^{\mathrm{T}}[A_{ll} - A_{ql}^{\mathrm{T}}A_{qq}^{-1}A_{ql} - (A_{\varepsilon l} - A_{\varepsilon q}A_{qq}^{-1}A_{ql} + \beta^{\mathrm{T}}A_{ql})^{\mathrm{T}} A_{cl}^{-1} \\ &\times (A_{\varepsilon l} - A_{\varepsilon q}A_{qq}^{-1}A_{ql} + \beta^{\mathrm{T}}A_{ql}) - 2\alpha^{\mathrm{T}}A_{\varepsilon q}]q_{,x} \end{aligned} \tag{64}$$

where A_{cl} is given by eqn (61).

The first and second summands in eqn (64) are positive definite. One can judge the positive definiteness of the whole expression by considering only the last summand. If appropriate entries of $A_{\varepsilon q}$ matrix are non-zero, it is always possible to make the last summand positive definite.

NUMERICAL RESULTS

In this section, we carry out the stiffness calculations and some one-dimensional analyses for three different configurations of prismatic beams (the effects related to initial curvature and twist are going to be present in a latter paper). Also, the corresponding eigenvalue problems are solved and some of the eigenmodes are considered in the stiffness calculation.

Note that a "classical" beam strain energy density can be obtained from eqn (55) by minimizing it with respect to the variables q with the derivatives $q_{,x}$ being set equal to zero. The result is an approximate strain energy per unit length of the form

$$2U^* = \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}^T [S] \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}. \tag{65}$$

Thus, the strain energy is in the same *form* as in a Euler–Bernoulli-like theory (the degrees of freedom of which are three sectional translations to accommodate extension and bending in two directions and one sectional rotation due to torsion). However, the appropriate coupling effects involving higher-order effects [such as transverse shear—see Hodges *et al.* (1992); Rehfield *et al.* (1990)] are present in eqn (65), and the numerical values of the resultant elastic constants can differ considerably from those of Euler–Bernoulli theory.

All the numerical results presented herein are based on six-node rectangular isoparametric element discretization ideally suited for composite beams. This element is incorporated in VABS [Variational–Asymptotical Beam Sectional Analysis by Cesnik (1994)] and the discretized model is submitted to the subroutine LANZ [by Jones and Patrick (1990)] an adaptation of the Lanczos algorithm, used to solve the large symmetric generalized eigenvalue problem. Then, the $(4+2N_q) \times (4+2N_q)$ stiffness matrix is used in a one-dimensional code based on a straightforward extension of the mixed-variational formulation of Hodges (1990b).

Isotropic case

In order to first check the characteristics of the above formulation, let us consider an isotropic beam with a square cross section of dimension 2 in by 2 in. The material properties are assumed to be $E = 2.6 \times 10^7$ psi and $\nu = 0.3$. We keep only the first two non-classical modes which are associated with an eigenvalue of multiplicity two. As we can see from Figs 2 and 3, these modes are predominantly out-of-plane rotation. The two eigenvalues are not exactly the same because of the asymmetry of the cross-sectional mesh. The discretization was done using a 10×20 six-node isoparametric element mesh. A first approximation of these modes would be rigid body out-of-plane rotations, associated with transverse shear effects. In fact, the shape of the curved side is the sine function, which is the exact solution for this problem for a square isotropic cross section.

It can be shown that the relationship between the new degrees of freedom associated with the above modes and the Timoshenko definition of transverse shear is given by

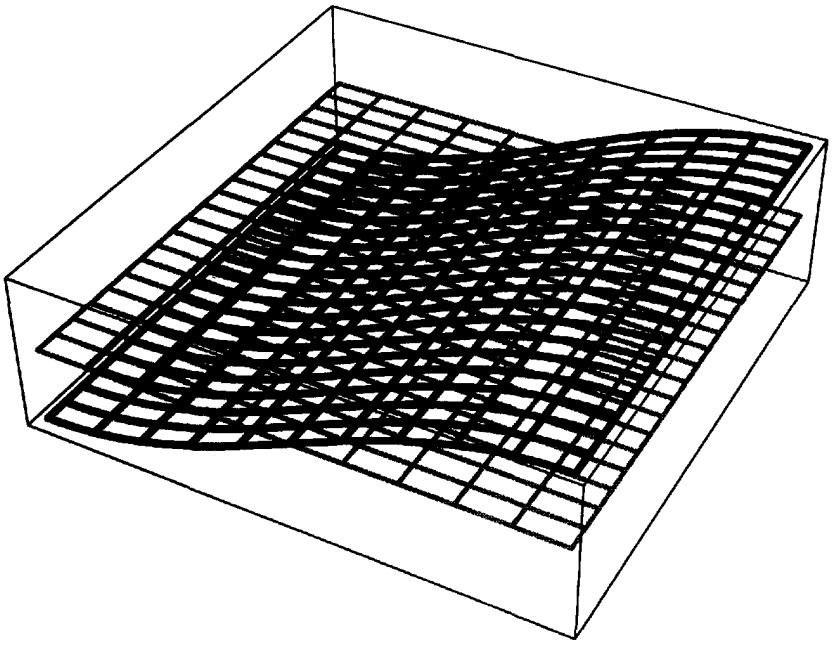

Fig. 2. Non-classical Mode 1 for the isotropic case ($\Lambda_1 = 2.467 \times 10^7$ psi).

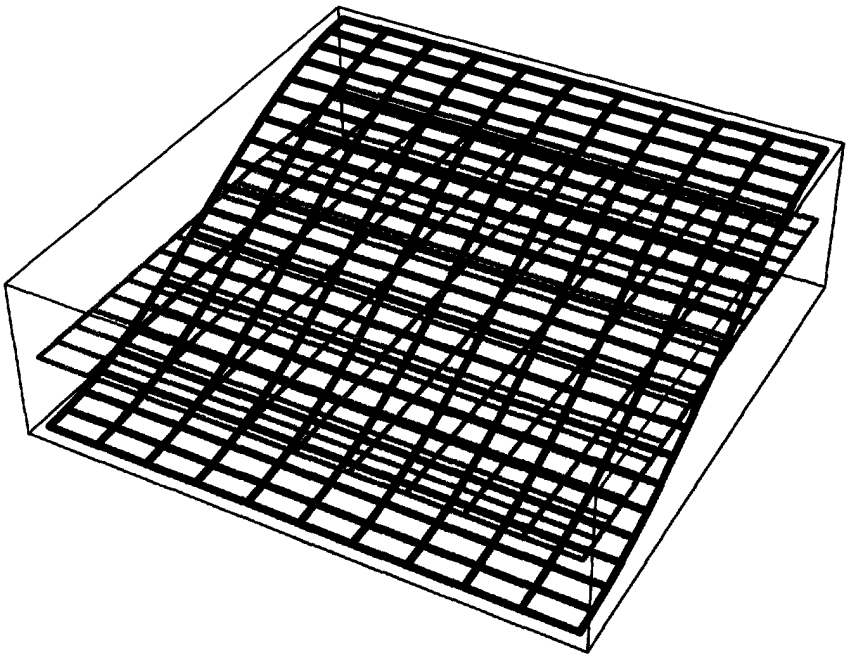

Fig. 3. Non-classical Mode 2 for the isotropic case ($\Lambda_2 = 2.472 \times 10^7$ psi).

$$2\gamma_{12} = -\frac{\langle x_2 \Psi_q \rangle}{I_3} q_1$$
$$2\gamma_{13} = -\frac{\langle x_3 \Psi_q \rangle}{I_2} q_2 \tag{66}$$

where I_α is the area moment of inertia. For the particular case under analysis, the numerical values are

$$2\gamma_{12} = -\frac{12}{\sqrt{2}\pi^2} q_1 \quad 2\gamma_{13} = -\frac{12}{\sqrt{2}\pi^2} q_2. \tag{67}$$

From the work of Berdichevsky and Staroselsky (1983), the asymptotically correct solution for the kind of cross section under consideration is given by

$$\begin{bmatrix} E|S| & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2^s G|S| & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3^s G|S| & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_3 \end{bmatrix}$$

with the shear correction factors as $k_\alpha^s = \frac{5}{6}$. Those results are asymptotically correct up to the second order, i.e. up to order h^2/l^2 in the strain energy.

Table 1. Stiffness results (lb, lb-in, and lb-in^2) for the squared isotropic cross section (1 extension; 2, 3 shear; 4 torsion; 5, 6 bending). Here, VABS$_q$ is the solution considering the new degrees of freedom q and VABS$_t$ is the previous one after undergoing the transformation described by eqn (66)

S	VABS$_q$	VABS$_t$	Berdichevsky and Staroselsky (1983)
S_{11}	1.040×10^8	1.040×10^8	1.040×10^8
S_{22}	2.467×10^7	3.338×10^7	3.333×10^7
S_{33}	2.472×10^7	3.345×10^7	3.333×10^7
S_{44}	2.252×10^7	2.252×10^7	2.249×10^7
S_{55}	3.468×10^7	3.468×10^7	3.467×10^7
S_{66}	3.467×10^7	3.467×10^7	3.467×10^7

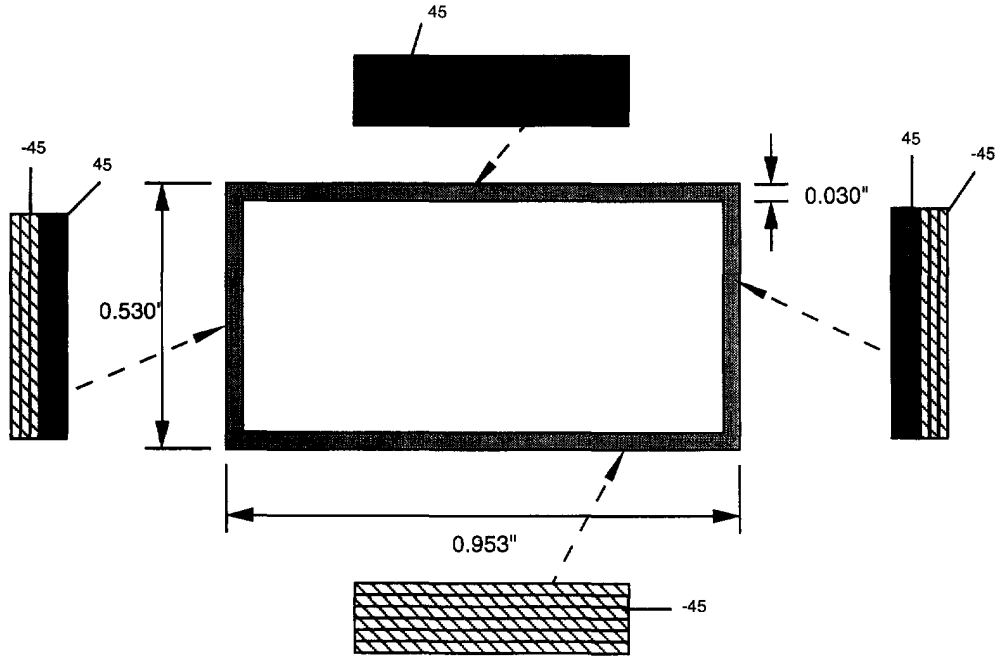

Fig. 4. Box beam cross section geometry and material.

In Table 1, the stiffness results for this example are shown and a direct comparison with the asymptotically correct solution of Berdichevsky and Staroselsky (1983) is presented. As one can see, the present approach gives us the correct transverse shear correction without introducing any *ad hoc* assumptions. The stiffness constants practically coincide with the ones that are asymptotically correct through order h^2/l^2. The slight difference between the two transverse shear stiffness constants is due to the asymmetric 6-node-element mesh used to discretize the cross section.

Composite box beam case

A box beam case was chosen among the experimental studies presented in Chandra *et al.* (1990). Its top and bottom are $[45^\circ]_6$ layups, and its sides are $[45^\circ]_{3a}$. The corresponding geometric and material properties are given in Fig. 4 and Table 2.

This case is a circumferentially asymmetric stiffness (CAS) configuration which produces bending–twist coupling. It is also known by the name symmetric configuration as adopted by Chandra *et al.* (1990), and Smith and Chopra (1991).

The cross section was discretized with 540 six-node isoparametric elements for a total of 1260 degrees of freedom. The stiffness matrix is reported in Table 3.

Table 2. Properties of the box beam cases (for material, note that the "L" direction is along the fibers and "N" is normal to the laminate)

Length = 30 in	Ply thickness = 0.005 in
$E_{LL} = 20.59 \times 10^6$ psi	
$E_{TT} = E_{NN} = 1.42 \times 10^6$ psi	
$G_{LT} = G_{LN} = 0.87 \times 10^6$ psi	
$G_{TN} = 0.70 \times 10^6$ psi	
$\nu_{LT} = \nu_{LN} = 0.42$	$\nu_{TN} = 0.50$

Table 3. Stiffness results (lb, lb-in, and lb-in^2) for the box beam case (1 extension; 2 torsion; 3, 4 bending)

$$\begin{bmatrix} 2.11 \times 10^5 & 0 & 0 & 0 \\ 0 & 2.07 \times 10^4 & 7.25 \times 10^3 & -3.61 \times 10^2 \\ 0 & 7.25 \times 10^3 & 1.22 \times 10^4 & -1.23 \times 10^2 \\ 0 & -3.61 \times 10^2 & -1.23 \times 10^2 & 2.81 \times 10^4 \end{bmatrix}$$

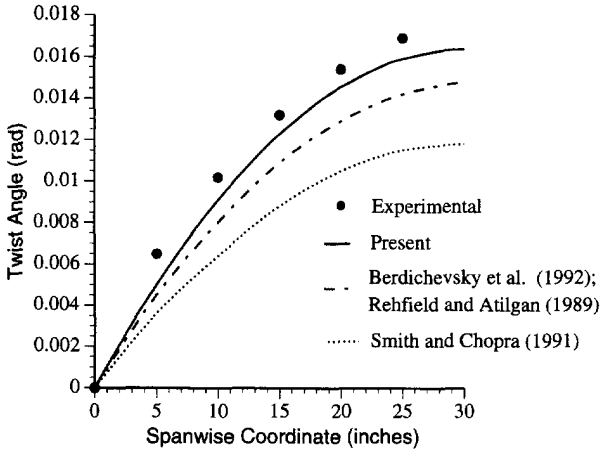

Fig. 5. Induced twist distribution along the beam length for the box beam case due to a unit vertical tip load.

In Fig. 5, the induced twist angle of a cantilever beam due to a unit vertical tip load is plotted against the spanwise coordinate. The present approach is compared with the experimental result and with the analytical predictions of Berdichevsky *et al.* (1992), Rehfield and Atılgan (1989), and Smith and Chopra (1990). As one can see, the correlation of the present approach with the experimental results is quite good. Even though the work of Berdichevsky *et al.* (1992) is based on a similar asymptotical approach, the thickness effect apparently prevents it from yielding better agreement with the experimental results. The complete thickness effect is included in the present numerical formulation. The inclusion of new degrees of freedom does not change this solution within plotting accuracy.

Composite I-beam case

For the open-section configuration, we have chosen a particular case from the experimental study done by Chandra and Chopra (1991). It is a bending–twist-coupled cantilever I-beam subject to a unit twisting moment applied at the free end. The cross section is made with graphite/epoxy material and its geometry is described in Fig. 6. The material properties are given in Table 4. The cross section was discretized with 590 six-node isoparametric elements for a total of 1277 degrees of freedom.

The eigen-analysis is performed first in order to get the most dominant mode associated with torsion. The "influence coefficient," defined in eqn (59), indicates that mode 11 is the

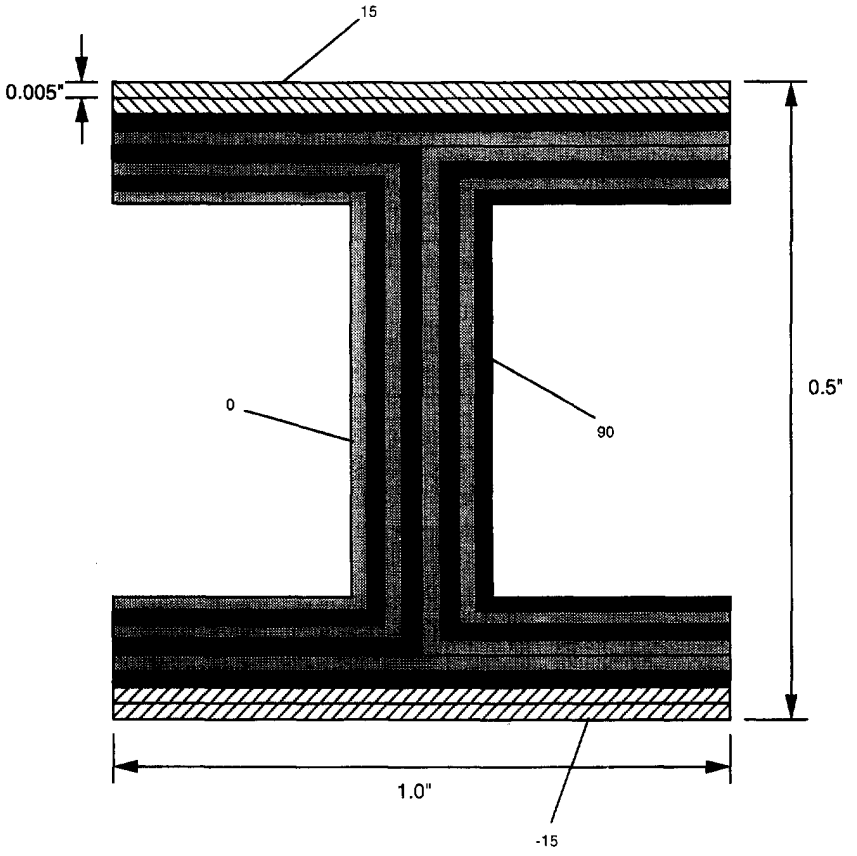

Fig. 6. I-beam cross section geometry and material (length = 30 in).

Table 4. Properties of the I-beam case (for material, note that the "L" direction is along the fibers and "N" is normal to the laminate)

$E_{LL} = 20.59 \times 10^6$ psi
$E_{TT} = E_{NN} = 1.42 \times 10^6$ psi
$G_{LT} = G_{LN} = G_{TN} = 0.87 \times 10^6$ psi
$\nu_{LT} = \nu_{LN} = \nu_{TN} = 0.42$

most dominant one, represented in Fig. 7. This mode matches what one expects from engineering judgment based on a Vlasov-type analysis [see Vlasov (1961)].

Considering only one extra degree of freedom q, we obtain the 6×6 stiffness matrix given in Table 5.

As one can see, the resulting stiffness matrix is non-positive definite ($A_{II} < 0$). By employing the transformation of variables described before, and using $\alpha = \lfloor 0 \ -62.76 \ 0 \ 0 \rfloor^T$ and $\beta = \lfloor 0 \ 0 \ 0 \ 0 \rfloor$, we are able to rewrite the stiffness matrix. The only change happens at the A_{II} term, which now reads

$$A_{II} = 9.0 \times 10^6.$$

With the corrected stiffness matrix, the present theory predicts a behavior which is in good agreement with the experimental results from Chandra and Chopra (1991) (see Fig. 8). The other analytical solutions shown in Fig. 8 are from Badir *et al.* (1993), who also use the variational–asymptotical approach, and Chandra and Chopra (1991). Both of them are still different from the experimental results. The classical stiffness matrix is unable to reproduce such behavior.

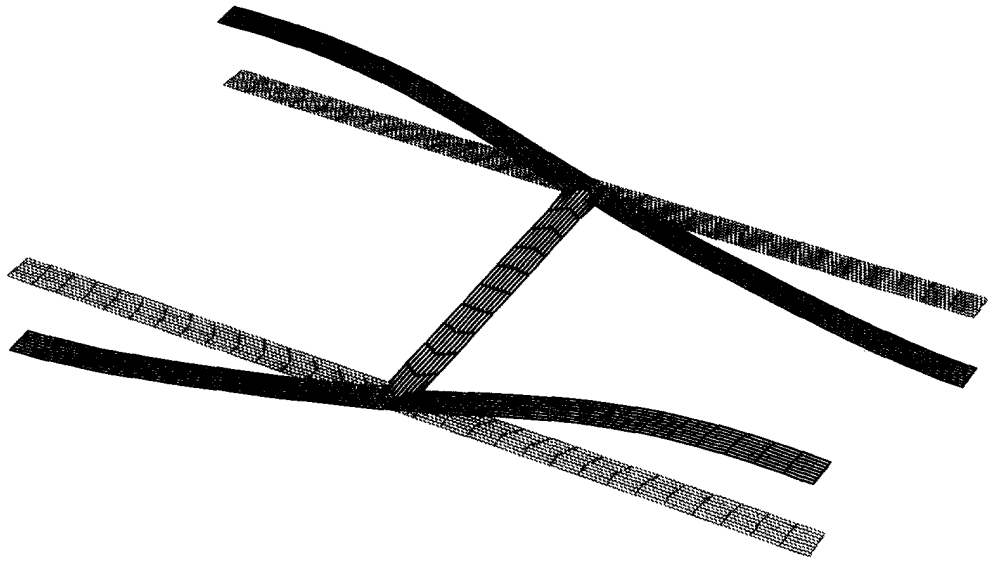

Fig. 7. Non-classical Mode 11 for the composite I-beam ($\Lambda_{11} = 1.110 \times 10^7$ psi, $\mathscr{I}_{qc}^{\text{torsion}} = 1.971 \times 10^{-2}$).

Table 5. Stiffness results (lb, lb-in, and lb-in^2) for the I-beam (1 extension; 2 torsion; 3, 4 bending; 5 q; 6 $q_{,x}$)

1.24×10^6	-1.70×10^2	2.47×10^4	-5.22×10^2	-2.05×10^4	-9.47×10^3
-1.70×10^2	4.44×10^3	4.49×10^3	0	2.22×10^5	3.04×10^2
2.47×10^4	4.49×10^3	6.26×10^4	0	2.04×10^5	3.17×10^3
-5.22×10^2	0	0	8.73×10^4	-1.38×10^1	7.73×10^4
-2.05×10^4	2.22×10^5	2.04×10^5	-1.38×10^1	1.13×10^7	0
-9.47×10^3	3.04×10^2	3.17×10^3	7.73×10^4	0	-1.88×10^7

CONCLUDING REMARKS

A refined beam theory containing the so-called classical sectional degrees of freedom plus an arbitrary number of "new" sectional degrees of freedom is presented. Evidently, the condition of asymptotical correctness is not sufficient to obtain a unique refined theory

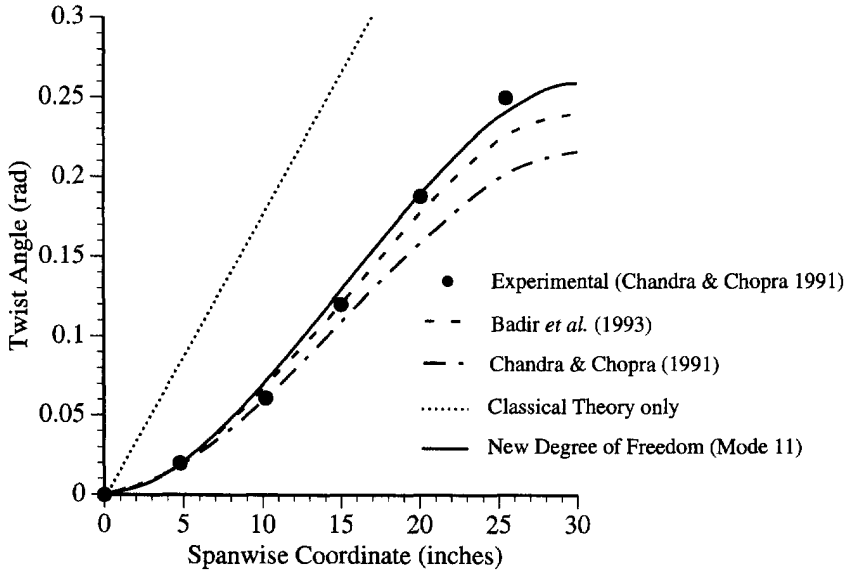

Fig. 8. Twist distribution along the beam length for I-beam case due to a unit tip torsional load.

of higher order than classical (Euler–Bernoulli-like) theory. Indeed, the present theory is asymptotically correct, but it can still exhibit a non-positive-definite strain energy density. This and other anomalies which might be present in an asymptotically correct theory developed with the long-wavelength hypothesis can be corrected by imposing different short-wavelength extrapolations, such as by changing the meaning of the variables of the one-dimensional theory. It is shown that a refined beam theory, one which appropriately possesses a positive-definite strain energy density and agrees closely with experimental results, can be constructed by using an appropriate short-wavelength extrapolation.

In principle this beam theory is capable of approximating three-dimensional elasticity to any accuracy desired. Indeed, asymptotically correct transverse shear stiffnesses are derived without *ad hoc* approximations. Numerical results for the stiffness constants of various composite beams are also presented which provide excellent predictive capability in the one-dimensional theory. Excellent agreement is obtained for some cases without adding any new sectional degrees of freedom, but another case, with an open cross section, is presented where at least one new degree of freedom is required to provide even reasonable correlation with experimental data.

The way short-wavelength extrapolations are constructed is an open question. Our hypothesis, that such extrapolations should result in good agreement between the one- and three-dimensional dispersion curves, needs to be confirmed with numerical experimentation.

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APPENDIX

The matrix E_u^+ (as well as E_{cl}^+) is used in the theoretical development as a way to get the solution for a linear system of the form

$$EV = (I - \bar{\Psi})b \tag{A1}$$

where $\bar{\Psi} = H\Psi_u\Psi_u^T$, i.e.

$$V = E_u^+ b. \tag{A2}$$

Equation (A1) is a generalization of the systems defined in eqns (45) and (50). Since the right-hand side vector is orthogonal to the null space of E (because of the definition of Ψ_u and H), the existence of V is guaranteed.

In order to check the relations stated in eqn (52), let us assume that all eigenvectors (Ψ) and eigenvalues of matrix E are known, i.e.

$$E\Psi = H\Psi\Lambda$$
$$\Psi^T H\Psi = I \tag{A3}$$

where the diagonal matrix of eigenvalues Λ is

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda_{N_q} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \lambda_{N_q-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \lambda_{N-4} \end{bmatrix}$$

and where N is the dimension of the matrix E.

Also, the following relations follow from eqn (A3)

$$\Psi\Psi^T = H^{-1}$$
$$E = H\Psi\Lambda\Psi^T H. \tag{A4}$$

Now, the matrix E_u^+ in eqn (52) can be defined as

$$E_u^+ = \Psi\Lambda_r^{-1}\Psi^T \tag{A5}$$

where the matrix Λ_r^{-1} is given by

$$\Lambda_r^{-1} \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{\lambda_{N_q-1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{\lambda_{N-4}} \end{bmatrix}.$$

The relations in eqn (52) are identically satisfied by just substituting eqns (A3)–(A5) into them.

However, due to the size of the matrices involved, a feasible numerical approach must be used. The procedure is divided in two parts. First, apply four point constraints, as in Atılgan *et al.* (1991) and Giavotto *et al.* (1983), in order to get the first part of the solution ($\hat{V}$). This will be off from the real solution V by the four rigid body modes (i.e. Ψ_{cl}). The reduced system is given by

$$\hat{E}\hat{V} = (\hat{I} - \hat{\Psi})b \tag{A6}$$

where ($\hat{}$) is ( ) after the four point constraints have been imposed.

Recover the full size (N) of the vector $\hat{V}$ (denoted here by $\hat{V}_{\text{ext}}$) by adding zeros to the corresponding removed degrees of freedom. So, the full solution can be written as

$$V = \hat{V}_{\text{ext}} + \Psi_{cl} c \tag{A7}$$

where c is a 4×1 column matrix which is determined by use of eqn (24)

$$c = -\Psi_{cl}^{\mathrm{T}} H \hat{V}_{\text{ext}}. \tag{A8}$$

This leads to the final expression for the solution of eqn (A1)

$$V = (I - \Psi_{cl}\Psi_{cl}^{\mathrm{T}} H)\hat{V}_{\text{ext}}. \tag{A9}$$

More details of this procedure can be found in Cesnik (1994).